INTRODUCTION TO CORPUS-BASED STATISTICS-ORIENTED (CBSO) TECHNIQUES

(PART II: BASIC CONCEPTS)

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CONTENTS

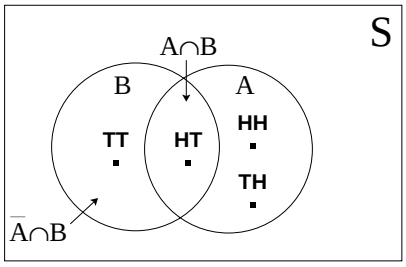
INTRODUCTION TO PROBABILITY

Basic Terminology

☐ Experiment:
- The process of observing a phenomenon that has variation in its outcomes.
- Example:
  Observing the outcomes of tossing a fair coin twice.

☐ Sample Space:
- The totality of the possible outcomes of a random experiment.
- Example:
  The sample space S={HH, HT, TH, TT}, where H: head; T: tail.

☐ Event:
- An event is a subset of the sample space.
- Example:
  A: at least one head in the two tosses.
  B: tail at the second toss.
  A={HT,TH, HH}, $\bar{A}$={TT}, B={HT,TT}



$$P(A) = \frac{3}{4},\ P(\bar{A}) = \frac{1}{4},\ P(B) = \frac{1}{2}$$
$$P(A \cap B) = \frac{1}{4},\ P(\bar{A} \cap B) = \frac{1}{4}$$
$$P(A \mid B) = \frac{1/4}{1/2} = \frac{1}{2}$$
$$P(\bar{A} \mid B) = \frac{1/4}{1/2} = \frac{1}{2}$$
$$P(B \mid A) = \frac{1/4}{1/3} = \frac{3}{4}$$
$$P(B \mid \bar{A}) = \frac{1/4}{1/4} = 1$$

Probability of an Event

☐ The probability of an event expresses the long-run frequency for the event occurring in many repeated experiments.

☐ Example:

$$P(A) = \frac{3}{4},\ P\left(\bar{A}\right) = \frac{1}{4},\ P(B) = \frac{1}{2}.$$

Probability of a Joint Event

☐ The probability of the joint event A and B is $P(A,B)$ (or $P(A \cap B)$)

☐ Example:

$A \cap B$={HT}, $P(A \cap B) = \frac{1}{4}$

$\bar{A} \cap B$={TT}, $P\left(\bar{A} \cap B\right) = \frac{1}{4}$

Conditional Probability

☐ The conditional Probability of the event A given that B event has occurred:

$$P(A \mid B) = \frac{P(A,B)}{P(B)}$$

☐ Example:

$$P(A \mid B) = \frac{1/4}{1/2} = \frac{1}{2}$$
$$P\left(\bar{A} \mid B\right) = \frac{1/4}{1/2} = \frac{1}{2}$$
$$P(B \mid A) = \frac{1/4}{1/3} = \frac{3}{4}$$
$$P\left(B \mid \bar{A}\right) = \frac{1/4}{1/4} = 1$$

☐ A and B are independent if and only if

$$P(AB) = P(A) \times P(B)$$
$$\Rightarrow P(A|B) = P(A)$$

Multiplication Theorem of Probability

☐ Theorem:

$$\begin{aligned} P(A,B) &= P(A \mid B) \times P(B) \\ &= P(B \mid A) \times P(A) \end{aligned}$$

☐ Example:

$$P(A,B) = \frac{1}{4}$$
$$P(B) \times P(A \mid B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$
$$P(A) \times P(B \mid A) = \frac{3}{4} \times \frac{1}{3} = \frac{1}{4}$$

☐ Generalization:

$$\begin{aligned} P(A_1, A_2, \cdots, A_k) = & P(A_1 \mid A_2, \cdots, A_k) \\ & \times P(A_2 \mid A_3, \cdots, A_k) \\ & \cdots \\ & \times P(A_{k-1} \mid A_k) \times P(A_k) \end{aligned}$$

Bayes' Rule

$$\begin{aligned} P(A \mid B) &= \frac{P(A,B)}{P(B)} = \frac{P(A,B)}{P(A,B) + P\left(\bar{A},B\right)} \\ &= \frac{P(A) \times P(B \mid A)}{P(A) \times P(B \mid A) + P\left(\bar{A}\right) \times P\left(B \mid \bar{A}\right)} \end{aligned}$$

☐ Example:

$$P\left(\bar{A}\right) = 1 - P(A) = \frac{1}{4}$$
$$P\left(B \mid \bar{A}\right) = \frac{P\left(B,\bar{A}\right)}{P\left(\bar{A}\right)} = \frac{1/4}{1/4} = 1$$
$$P(A \mid B) = \frac{\frac{3}{4} \times \frac{1}{3}}{\left(\frac{3}{4} \times \frac{1}{3}\right) + \left(\frac{1}{4} \times 1\right)} = \frac{1}{2}$$

☐ Generalization:

$$P(A_k \mid B) = \frac{P(B \mid A_k) \times P(A_k)}{\sum\limits_{A_i} P(B \mid A_i) \times P(A_i)}$$

where $A_1, A_2, \cdots, A_n$ are partitions of the sample space; i.e.,

$$A_1 \cup A_2 \cup \cdots \cup A_n = S,$$
$$A_i \cap A_j = \phi\ \forall\ i \neq j.$$

Discrete Random Variable

☐ A random variable X on a sample space S is a function $X : S \to \mathrm{R}$ that assigns a real number $X(s)$ to each sample point $s \in S$.

☐ Example:
X: number of head in the two tosses,
$[X = 0]$={TT}
$[X = 1]$={TH,HT}
$[X = 2]$={TT}.

Continuous Random Variable

☐ A random variable X on a probability space $(S, \mathcal{F}, P)$ is a function $X : S \to \mathrm{R}$ that assigns a real number $X(s)$ to each sample point $s \in S$, such that for every real number x, the set $\{s | X(s) \leq x\}$ is an event, where $\mathcal{F}$ denotes the class of measurable subsets of S.

Distribution Function

☐ The distribution function F_X of a random variable X is defined to be a function

$$F_X(x) = P(X \leq x), \ -\infty < x < \infty.$$

☐ Example: the continuous uniform distribution:

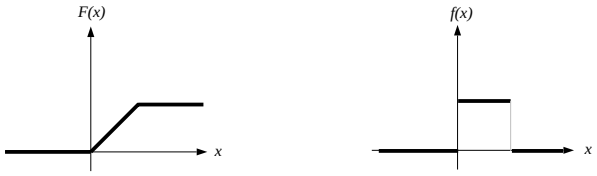
$$F_X(x) = \begin{cases} 0, & x < 0, \\ x, & 0 \leq x < 1, \\ 1, & x \geq 1. \end{cases}$$

Probability Density Function

☐ For a continuous random variable, X, $f(x) = \frac{dF_X(x)}{dx}$ is called the probability density function (pdf) of X

☐ Example: the continuous uniform distribution:

$$f(x) = \begin{cases} 0, & x < 0, \\ 1, & 0 \leq x < 1, \\ 0, & x \geq 1. \end{cases}$$



Mean

☐ The measure of central tendency or expected value of a random variable.

☐ A weighted average of the possible values of the random variable.

$$\mu_X = E[X] = \begin{cases} \sum\limits_{x_i} x_i P(x_i), & (discrete) \\ \int\limits_{-\infty}^{\infty} x f(x)\, dx, & (continuous). \end{cases}$$

Variance

☐ The measure of dispersion for a random variable.

☐ A weighted average which indicates how much individual values differ from the center of the distribution.

$$Var(X) = E\left[(X - \mu_X)^2\right] = \begin{cases} \sum\limits_{x_i} (x_i - \mu_X)^2 P(x_i), & (discrete) \\ \int\limits_{-\infty}^{\infty} (x - \mu_X)^2 f(x)\, dx, & (continuous). \end{cases}$$

Special Discrete Distributions

☐ **Bernoulli Distribution**:

- A random variable X has the Bernoulli distribution if (for some $p, (0 \leq p \leq 1)$)

$$P(X = x) = \begin{cases} p^x (1-p)^{1-x}, & x = 0, 1 \\ 0, & otherwise. \end{cases}$$

- A Bernoulli random variable X can be interpreted as the number of successes in one trial of an experiment where the probability of success is p.
- $E[X] = p; \ \ Var[X] = p(1-p)$.

☐ **Binomial Distribution**:

- A random variable X has the binomial distribution if (for some integer n, and some $p, (0 \leq p \leq 1)$)

$$P(X = x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x = 0, 1, \cdots, n \\ 0, & otherwise. \end{cases}$$

- A binomial random variable can be considered as a sum of n Bernoulli random variables, that is as the number of successes in n Bernoulli trials.
- When sampling from finite populations, the binomial distribution arises only when the sampling is done with replacement.
- $E[X] = np; \ \ Var[X] = np(1-p)$.

☐ **Multinomial Distribution**:

- Consider an experiment consisting of n independent and identical trials, in which each trial can result in any one of r possible outcomes.
- Let random variables X_i denote the number of trials resulting in outcome i $(i = 1, \cdots, r)$. The joint distribution of $X_1, \cdots, X_r$ has the *multinomial distribution*:

$$P(x_1, \cdots, x_r) = \begin{cases} \frac{n!}{(x_1!)(x_2!)\cdots(x_r!)} p_1^{x_1} p_2^{x_2} \cdots p_r^{x_r}, & x_i = 1, 2, \cdots, and \sum_{i=1}^{r} x_i = n, \\ 0, & otherwise. \end{cases}$$

☐ **Poisson Distribution**:

- A random variable X has the Poisson distribution if (for some $\mu > 0$, called a *parameter* of the distribution)

$$P(X = x) = \begin{cases} e^{-\mu} \frac{\mu^x}{x!}, & x = 0, 1, \cdots \\ 0, & otherwise. \end{cases}$$

- The parameter μ can be interpreted as the "average" number of occurrences of the event)
- $E[X] = Var[X] = \mu$.

Special Continuous Distribution

☐ **Normal Distribution (Gaussian Distribution)**:

- A random variable X has the normal distribution $N(\mu, \sigma^2)$ if (for some $\sigma^2 > 0$ and $-\infty < \mu < \infty$)

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \ -\infty < x < \infty.$$

- $N(0, 1)$ distribution is called the *standard normal* distribution.
- $E[X] = \mu; \ \ Var[X] = \sigma^2$.

☐ **Chi-Square Distribution**:

- A random variable X has the chi-square distribution with ν degrees of freedom (for some $\nu \in \mathbf{N}$)

$$f_X(x) = \begin{cases} \frac{1}{2^{\nu/2}\Gamma\left(\frac{\nu}{2}\right)} x^{(\nu/2)-1} e^{-x/2}, & x > 0 \\ 0, & otherwise. \end{cases}$$

- Let $X_1, X_2, \cdots, X_\nu$ be ν i.i.d. random variables with p.d.f. $N(0, 1)$, the random variable

$$\chi^2 = X_1^2 + X_2^2 + \cdots + X_\nu^2$$

has a chi-square distribution with ν degrees of freedom.

- $E[X] = \nu; \ \ Var[X] = 2\nu$.

☐ **t-distribution**

- A random variable X has the t-distribution with n degrees of freedom (for some integer $n > 0$)

$$f_X(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)} \frac{1}{\left(1 + \frac{x^2}{n}\right)^{(n+1)/2}}, \quad -\infty < x < \infty.$$

- Properities:
  - — t-distribution curve is bell-shaped and centered at 0.
  - — As the degree of freedom n increases, the spread of the corresponding distribution curve decreases.
  - — Each t-distribution curve is more spread out than the standard normal curve.
  - — As the degree of freedom $n \to \infty$, the sequence of t-distribution curve approaches the standard normal curve.

Joint Distribution and Density

☐ **Joint Distribution**:

- The joint distribution $F(x,y)$ of two random variables X and Y is the probability of the event $\{X \leq x, Y \leq y\}$, i.e.,

$$F(x,y) = P(X \leq x, Y \leq y).$$

☐ **Joint Density**:

- The joint density $f(x,y)$ of two random variables X and Y is

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \, \partial y}.$$

Marginal Distribution and Density

☐ **Marginal Distribution**:

- The marginal distribution $F(x)$ of X is

$$F_X(x) = F(x,\infty).$$

- The marginal distribution $F_Y(y)$ of Y is

$$F_Y(y) = F(\infty,y).$$

☐ **Marginal Density**:

- The marginal density $f_X(x)$ of X is

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy.$$

- The marginal distribution $f_Y(y)$ of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx.$$

Conditional Distribution and Density

☐ **Conditional Distribution**:

- The conditional distribution $F(x|\mathcal{U})$ of the random variable X assuming $\mathcal{U}$ is

$$F(x|\mathcal{U}) = \frac{P(X \leq x, \mathcal{U})}{P(\mathcal{U})}.$$

☐ **Conditional Density**:

- The conditional density $f(x|\mathcal{U})$ of the random variable X assuming $\mathcal{U}$ is

$$f(x|\mathcal{U}) = \frac{dF(x|\mathcal{U})}{dx}.$$

Covariance and Correlation Coefficient

☐ **Covariance**:

- The covariance C_{xy} of two random variables X and Y is defined as follows:

$$C_{xy} = E[(X - \mu_X)(Y - \mu_Y)].$$

☐ **Correlation Coefficient**:

- The correlation coefficient ρ_{xy} of two random variables X and Y is defined as follows:

$$\rho_{xy} = \frac{C_{xy}}{\sigma_x \sigma_y}.$$

Independence vs. Uncorrelatedness

☐ **Independence**:

- The random variables X and Y are called *independent* if

$$f_{XY}(x,y) = f_X(x) f_Y(y)$$
$$\left(f_{X|Y}(x|y) = f_X(x)\right).$$

☐ **Uncorrelatedness**:

- The random variables X and Y are called *uncorrelated* if their covariance is zero, i.e.,

$$C_{xy} = 0,\ \rho_{xy} = 0,\ E[XY] = E[X]E[Y].$$

☐ If two random variables are *independent*, then they are *uncorrelated* (but vice versa).

☐ **Orthogonality**:

- The random variables X and Y are said to be *orthogonal* if

$$E[XY] = 0.$$

Random Vector

☐ A *random vector* is a vector $\mathbf{X} : [X_1, \cdots, X_n]$ whose components X_i are random variables.

☐ The probability that $\mathbf{X}$ is in a region D of the n-dimensional space is

$$P(\mathbf{X} \in D) = \int_D f(x_1, \cdots, x_n)\, dx_1 \cdots dx_n.$$

☐ The random variables $X_1, \cdots, X_n$ are (mutually) independent if the events $\{X_1 \le x_1\}, \cdots, \{X_n \le x_n\}$ are independent, i.e.,

$$F(x_1, \cdots, x_n) = F(x_1) \cdots F(x_n)$$
$$f(x_1, \cdots, x_n) = f(x_1) \cdots f(x_n).$$

Let $X_1, X_2, \cdots X_n$ be *normally distributed* random variables having means $\mu_1, \mu_2, \cdots, \mu_n$ and variances $\sigma_1, \sigma_2, \cdots, \sigma_n$, respectively. Let $a_1, a_2, \cdots, a_n$ be constants. Then the random variable Y which is the linear combination of the $X's$, i.e., $Y = a_1X_1 + a_2X_2 + \cdots + a_nX_n$ is also normally distributed with mean μ_Y and variance σ_Y^2, where

$$\mu_Y = a_1\mu_1 + a_2\mu_2 + \cdots + a_n\mu_n$$
$$\sigma_Y^2 = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \cdots + a_n^2\sigma_n^2.$$

INTRODUCTION TO STATISTICS

Basic Terminology

☐ **Point Estimate**:

- A *point estimate* is a single number that is used as an estimate of a population parameter or population characteristic.
- Example:
  - — Head appears 15 time in 25 independent tosses, then the estimated probability for appearing head is $\hat{p} = \frac{15}{25} = 0.6$.

☐ **Interval Estimate**:

- An *interval estimate* is an interval that provides an upper and lower bound for a specific population parameter whose value is unknown.
- This interval estimate has an associated degree of confidence of containing the population parameters. Such interval estimates are also called **confidence intervals**.
  - — e.g., $\mu = \bar{x} \pm 0.03$ with 95% confidence interval

$$P(|\mu - \bar{x}| \le 0.03) \ge 0.95.$$

☐ **Estimators**:

- An *estimator* is a random variable calculated from sample data that provides either point estimates or interval estimates for some population parameter.
- **Unbiasedness**:
  - — An estimator $\hat{\theta}$ is *unbiased* if its mean is equal to the population parameter being estimated θ, i.e., $E\left[\hat{\theta}\right] = \theta$.
- **Efficiency**:
  - — An estimator $\hat{\theta}$ of θ is said to be more *efficient* than any other unbiased estimator $\hat{\hat{\theta}}$ if $Var\left(\hat{\theta}\right) \le Var\left(\hat{\hat{\theta}}\right)$.
  - — An estimator is a **minimum variance unbiased estimator** if the variance of its sampling distribution is the smallest of all other unbiased estimators.
- **Consistency**:
  - — An estimator is said to be a *consistent estimator* if it approaches the parameter to be estimated *in a probability sense* as the sample size n gets large, i.e.,

$$\lim_{n\to\infty} P\left(\left|\hat{\theta}(n) - \theta\right| \ge \epsilon\right) = 0,$$

where ϵ is a small positive number.

Maximum Likelihood Estimation

☐ To choose a set of parameters θ in a way that maximizes the likelihood function $L(\theta)$:

$$L(\theta) = f(x_1, x_2, \cdots, x_n|\theta) = \prod_{i=1}^{n} f(x_i|\theta),$$

where $x_1, x_2, \cdots, x_n$ is a set of random samples from the distribution of a random variable X with density f and associated parameter θ.

☐ The ML estimation $\hat{\theta} = \left(\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_k\right)$ is the set of estimated values that satisfies the equations

$$\frac{\partial L(\theta)}{\partial \theta_i} = 0,\ i = 1, \cdots, k.$$

☐ **Properties**:

- Maximum-likelihood estimates are (1) consistent, and (2) asymptotically efficient.
- Let $\hat{\theta}_{ML}$ be a MLE of θ, then $g\left(\hat{\theta}_{ML}\right)$ is a MLE of $g(\theta)$, i.e.,

$$\widehat{\left[g(\theta)\right]}_{ML} = g\left(\hat{\theta}_{ML}\right),$$

where $g(\cdot)$ is a monotonic function.

☐ Examples:

- The MLE for the "success" probability p of Bernoulli distribution is

$$\hat{p}_{ML} = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i,$$

where $x_i\ (= 0\ or\ 1)$ is the outcome of the i-th Bernoulli trial.

— $\hat{p}_{ML}$ can be interpreted as the relative frequency of success over the n trials.

- The MLEs for the mean and variance of the normal density are:

$$\hat{\mu}_{ML} = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i,$$
$$\sigma^2_{ML} = \frac{1}{n}\sum_{i=1}^{n} (x_i - \bar{x})^2.$$

Bayesian Estimation

☐ To choose the parameters which maximizes the likelihood function $L(\pi(\theta), \theta)$:

$$L(\pi(\theta), \theta) = \pi(\theta) f(x_1, \cdots, x_n|\theta),$$

where $\pi(\theta)$ is the prior probability density of θ before sampling.

☐ The Bayesian estimation $\hat{\theta} = \left(\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_k\right)$ is the set of estimated values that satisfies the equations

$$\frac{\partial L(\pi(\theta), \theta)}{\partial \theta_i} = 0,\ i = 1, \cdots, k.$$

☐ The Bayesian estimation $\hat{\theta}$ of parameter θ is the expected value of the parameter taken with respect to the *posterior distribution* of θ given the outcome of the random sample x , i.e.,

$$\hat{\theta} = E[\theta \mid x].$$

☐ Example:

- To estimate the mean μ of a normal density $N(\mu, \sigma^2)$ with the known value of σ. Let $\pi(\theta) \sim N(\mu_0, \alpha)$, then the Bayesian estimate $\hat{\mu}_{Baye}$ of μ is

$$\hat{\mu}_{Baye} = \frac{\sigma^2\mu_0 + \alpha^2 n\bar{x}}{\sigma^2 + \alpha^2 n},$$

— n is the number of samples,
— $\bar{x}$ is the sample mean.

Least Squares Estimation

☐ To estimate θ by the point $\tilde{\theta} = \left(\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_k\right)$ which makes the corresponding expected values $g_1(\theta), g_2(\theta), \cdots, g_n(\theta)$ as close as possible to the observed samples $x_1, x_2, \cdots, x_n$.

☐ The LS estimation $\tilde{\theta} = \left(\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_k\right)$ is the set of estimated values that satisfies the equations

$$x_i = g_i(\theta) + \epsilon_i,\ 1 \le i \le n,$$
$$\frac{\partial}{\partial \theta_j}\sum_{i=1}^{n}\left[x_i - g_i(\theta)\right]^2 = 0,\ j = 1, \cdots, k.$$

Hypothesis Testing

☐ **Goal**:
To make a binary decision on a hypothesis based on the given observations.

☐ **Hypotheses**:

- **Null Hypothesis** (H_0):
The hypothesis that we are interested in rejecting or refuting.
- **Alternative Hypothesis** (H_1):
The contradictory hypothesis of H_0.

☐ **Decision Regions**:
The observation space is partitioned into acceptance region R(H_0) and rejection region R(H_1); if the observed features fall within the acceptance region, hypothesis H_0 is confirmed, otherwise, H_0 is rejected.

☐ **Types of errors**:

- **Type I error**:
H_0 is true but the observation suggests H_1
- **Type II error**:
H_0 is false but the observation suggests H_0.

☐ **Level of Significance**:
The *level of significance*, denoted by α, is the maximum probability of making a *Type I error*.

☐ **One-Tailed Test vs. Two-Tailed Test**:
For a test statistic T computed on the sample data:

- a *upper one-tailed test* has the decision rule:
— Reject H_0 if $T > T_U$; otherwise accept H_0.
- a *lower one-tailed test* has the decision rule:
— Reject H_0 if $T < T_L$; otherwise accept H_0.
- a *two-tailed test* has the decision rule:
— Reject H_0 if $T > T_U$ *or* $T < T_L$;
Accept H_0, otherwise.
- The values of T_U and T_L are critical values that are selected so that the test will have the desired level of significance α.

☐ **p-Value**:
The *p-value* associated with a test statistic is the *smallest* level of significance that would have allowed the null hypothesis to be rejected.

☐ **Procedures**:

1. State the null hypothesis, H_0.
2. State the alternative hypothesis , H_1.
3. Decide on the level of significance, α.
4. Choose an appropriate testing procedure and determine the acceptance region.
5. Compute the test statistic from the sample data.
6. Make the decision: reject H_0 if the p-value is less than the level of significance α; otherwise accept H_0.

☐ Example:

To test H_0: $p = 0.6$ against H_1: $p \neq 0.6$.

For a two-tailed test of the level of significance $\alpha = 0.05$, the critical values of the normal distribution are $T_U = 1.96;\ T_L = -1.96$.

Suppose the computed test statistic $T = 2.06$ which corresponds to the p-value of 0.0394.

We will accept H_0 of the level of significance $\alpha = 0.05$.

However, we will reject H_0 if the level of significance $\alpha = 0.01$.

Likelihood Ratio Test

☐ The **likelihood ratio** λ:

$$\lambda = \frac{f_0(X)}{f_1(X)},$$

where

- H_0: the pdf of the data is $f_0(X)$,
- H_1: the pdf of the data is $f_1(X)$.

Accept H_0 if $\lambda > \lambda_T$ (λ_T is a preset threshold), otherwise accept H_A.

☐ Example:
For automatic compound noun extraction [Su 94]:

- H_0: the feature vector $\vec{x}$ for the input pattern is generated by a compound model M_c.
- H_A: the feature vector for the input pattern is generated by a *non-compound* model M_{nc}.

the likelihood ratio λ is

$$\lambda = \frac{P(M_c \mid \vec{x})}{P(M_{nc} \mid \vec{x})}$$

INTRODUCTION TO STOCHASTIC PROCESS

☐ A *stochastic process* $\{X(t), t \in T\}$ is a collection of random variables; i.e, for each $t \in T$, $X(t)$ is a random variable.

☐ Interpretations:

- A stochastic process $X(t) = X(t, \zeta)$ is a single time function (a sample of the given process) if ζ is fixed.
- $X(t, \zeta)$ becomes a random variable equal to the *state* of the given process at time t, if t is fixed.
- If t and ζ are fixed, then $X(t, \zeta)$ is a number.

☐ The set T is called the *index* set of the process.

- $\{X(t)\}$ is a *discrete-time* process, if T is a countable set; e.g., $\{X_n, n = 0, 1, \cdots\}$.
- $\{X(t)\}$ is a *continuous-time* process, when T is an interval of the real line; e.g., $\{X(t), t \geq 0\}$.

☐ Example:

- $\{X(t)\}$ might be equal to the total number of customers that have entered a supermarket by time t.

Markov Chains

☐ A discrete-time discrete-state stochastic process $\{X_n, n = 0, 1, \cdots\}$, having the property that given the present state, the past states have no influence on the future, is called a discrete-time *Markov chain*

☐ The Markov property:

$$P(X_n = j \mid X_{n-1} = i, X_{n-2} = i_{n-2}, \cdots, X_0 = i_0) = P(X_n = j \mid X_{n-1} = i),$$

- $P(X_n = j \mid X_{n-1} = i)$ are called the *transition probabilities* of the chain.

☐ Example:
The formula for tagging part-of-speech is approximated as:

$$\max \prod_{i=1}^{n} P(w_i \mid t_i) \cdot P(t_i \mid t_{i-1}),$$

where t_i corresponds to the part-of-speech attached to the i-th word w_i.

- The probability $P(t_i \mid t_{i-1})$ in the above formula is the transition probability of the assumed Markov model.

INTRODUCTION TO INFORMATION THEORY

Entropy

Let each possible outcome x_k of a stationary source X occur with a probability of $P(x_k)$.

☐ **Self-information** $I(x_k)$:

$$I(x_k) = -\log P(x_k)$$

- $I(x_k)$ is the amount of information associated with the known occurrence of output x_k.

☐ **Entropy** $H(X)$:

$$H(X) = -\sum_i [P(x_i) \cdot \log P(x_i)]$$

- $H(X)$ is the average information (or uncertainty) of the source X.

Mutual Information

☐ **Mutual Information** $I(x; y)$:

$$I(x; y) \equiv \log_2 \left[\frac{P(x, y)}{P(x) \cdot P(y)}\right]$$

- $I(x; y)$ is the information that the reception of y supplies about x.

☐ Example: Use $I(w_x; w_y)$ as a measure for the preference of "strong economy" and "powerful economy" [K. Church 89]:

- $I(w_x; w_y) >> 0$, w_x and w_y are highly associated.
- $I(w_x; w_y) \approx 0$, w_x and w_y are independent.
- $I(w_x; w_y) << 0$, w_x and w_y are in complementary distribution.

Perplexity

- The perplexity is a measure of the constraint imposed by the grammar, or the level of uncertainty given the grammar.
- Let $P(w \mid s)$ be the probability that w will be next word when the current state is s.
  - The entropy, $H_s(w)$, associated with state s is

    $$H_s(w) = -\sum_w P(w \mid s) \log_2 P(w \mid s).$$

  - The entropy $H(w)$ of the task is the average value of $H_s(w)$, i.e.,

    $$H_s(w) = \sum_s \pi(s) H_s(w),$$

    where $\pi(s)$ is the probability of being in state s during the production of a sentence.
- The **perplexity** $S(w)$ of the task [Bahl 83]

  $$S(w) = 2^{H(w)}.$$